Prandtl-Meyer Function Web Application

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1 Introduction

The Prandtl-Meyer function is used to calculate the change in Mach number or flow inclination angle when a supersonic flow undergoes an isentropic expansion or compression by turning. Referring to Figure 1, an isentropic expansion by turning occurs when a gas flows over a convex corner, so that the flow along the wall is turned away from the main flow. An isentropic compression by turning occurs when a gas flows over a concave corner, so that the flow along the wall is turned into from the main flow.

In Figure 1 the Mach number $M$ and flow inclination angle $\theta$ change from $(M_1, \theta_1)$ to $(M_2, \theta_2)$. The Prandtl-Meyer function enables us to determine $M_2$ given $M_1, \theta_1$ and $\theta_2$, or $\theta_2$ given $M_1, \theta_1$ and $M_2$.

The function applies to a calorically perfect gas, and the derivation of the function can be found in textbooks on compressible flow, such as Refs. [1] and [2]. The function has the form

$$\theta = \nu(M)$$

The angle $\theta$ is chosen to be zero when $M = 1$ and increases monotonically with $M$. Evaluating $\nu(M)$ is laborious, and many textbooks, such as Refs. [1] and [2], give tables of $\nu(M)$ against $M$. The Prandtl-Meyer function web application is intended to replace these tables.

Figure 1 Isentropic expansion and compression by turning
1.1 Isentropic expansion

If the flow incidence angle $\theta_2$ after an isentropic expansion is known, then we can calculate the Prandtl-Meyer function at exit from the corner as follows:

$$v_2 = v_1 + |\theta_2 - \theta_1|$$

where $v_1 = v(M_1)$ and $v_2 = v(M_2)$. The exit Mach number $M_2$ can then be determined from $v_2$ using tables.

If the exit Mach number $M_2$ is known, then we calculate the exit incidence angle as follows:

$$|\theta_2 - \theta_1| = v_2 - v_1$$

The flow angle $\theta_2$ can take two values, but the turning is $v_2 - v_1$ in both cases.

1.2 Isentropic compression

If the flow incidence angle $\theta_2$ after an isentropic compression is known, then we can calculate the Prandtl-Meyer function at exit from the corner as follows:

$$v_2 = v_1 - |\theta_2 - \theta_1|$$

The exit Mach number $M_2$ can then be determined from $v_2$ using tables.

If the exit Mach number $M_2$ is known, then we calculate the exit incidence angle as follows:

$$|\theta_2 - \theta_1| = v_1 - v_2$$

The flow angle $\theta_2$ can take two values, but the turning is $v_1 - v_2$ in both cases.
2 Mach number from the Prandtl-Meyer function

The Prandtl-Meyer function is

\[ \nu(M) = \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left( \frac{\gamma - 1}{\sqrt{\gamma + 1} (M^2 - 1)} \right) - \tan^{-1} \left( \sqrt{M^2 - 1} \right) \] [radians] \hspace{1cm} (1)

where \( \gamma = c_p/c_v \) is the ratio of specific heats of the gas. Tabulated values of \( \nu(M) \) against \( M \) are usually for \( \gamma = 1.4 \), which is the ratio of specific heats of the International Standard Atmosphere (Ref. [3]).

The equation expresses \( \nu \) explicitly in terms of \( M \). The variation of \( \nu \) with \( M \) is shown in Figure 2. The ratio of specific heats \( \gamma \) in Figure 2 is 1.4.

**Figure 2 Prandtl-Meyer function**

As \( M \to \infty \) the arctan functions is Eqn. (1) tend to \( \pi/2 \) and so we can write

\[ \nu(M \to \infty) = \frac{\pi}{2} \left( \frac{\gamma + 1}{\sqrt{\gamma - 1} - 1} \right) \] [radians]

When \( \gamma = 1.4 \), \( \nu(M \to \infty) = 130^\circ \) (2.27 rad). This is the upper limit of \( \nu \) for which it is possible to find a value of \( M \).
To the author’s knowledge there is no general solution of the inverse of the Prandtl-Meyer function for any \(\gamma\). An exact solution has been published for \(\gamma = 5/3\) (see Ref. [4]). In this case, the square root term involving \(\gamma\) is particularly simple:

\[
\sqrt{\frac{\gamma + 1}{\gamma - 1}} = 2
\]

However, we can calculate the inverse quite quickly and easily for the general case by using a numerical method on a computer. The Newton-Raphson iterative method is well suited to the task.

In order to use the Newton-Raphson method, we require an equation for the derivative \(d\nu/dM\). We can write the Prandtl-Meyer function as follows.

\[
\nu = \frac{1}{\lambda} \tan^{-1}(\lambda \beta) - \tan^{-1}(\beta)
\]

where

\[
\lambda = \frac{\sqrt{\gamma - 1}}{\sqrt{\gamma + 1}}
\]

and

\[
\beta = \sqrt{M^2 - 1}
\]

Differentiating \(\nu\) with respect to \(\beta\),

\[
\frac{d\nu}{d\beta} = \frac{1}{\lambda} \frac{\lambda}{1 + \lambda^2 \beta^2} - \frac{1}{1 + \beta^2} = \frac{(1 - \lambda^2)\beta^2}{(1 + \beta^2)(1 + \lambda^2 \beta^2)}
\]

Differentiating \(\beta\) with respect to \(M\),

\[
\frac{d\beta}{dM} = \frac{M}{\sqrt{M^2 - 1}} = \frac{M}{\beta}
\]

By the chain rule,

\[
\frac{d\nu}{dM} = \frac{d\nu}{d\beta} \frac{d\beta}{dM} = \frac{(1 - \lambda^2)\beta^2}{(1 + \beta^2)(1 + \lambda^2 \beta^2)} \times \frac{M}{\beta} = \frac{(1 - \lambda^2)\beta}{M(1 + \lambda^2 \beta^2)} (2)
\]

since \(1 + \beta^2 = M^2\).
3 Newton-Raphson method

Using Eqn. (1) we can define the function

\[ f(M) = \frac{\gamma + 1}{\gamma - 1} \tan^{-1} \left( \frac{\gamma - 1}{\gamma + 1} (M^2 - 1) \right) - \tan^{-1} \left( \sqrt{M^2 - 1} \right) - \nu \quad (3) \]

The Mach number \( M \) is now the root of the function \( f(M) \). We can use the Newton-Raphson iterative method to find the root.

In the Newton-Raphson method we make an initial guess at the root, \( M_i \). We then draw a tangent from the point \([M_i, f(M_i)]\). The point where this tangent crosses the \( M \) axis usually represents an improved estimate \( M_{i+1} \) of the root.

The first derivative \( f'(M_i) \) at \( M_i \) is equivalent to the tangent to the point, so the new estimate \( M_{i+1} \) is given by:

\[ f'(M_i) = \frac{f(M_i) - 0}{M_i - M_{i+1}} \]

Rearranging this equation gives

\[ M_{i+1} = M_i - \frac{f(M_i)}{f'(M_i)} \]

which is called the Newton-Raphson formula.

The last term in Eqn. (3) for \( f(M) \) is a constant, so the first derivative, \( f'(M) \), is given by Eqn. (2),

\[ f'(M_i) = \frac{(1 - \lambda^2) \beta}{M(1 + \lambda^2 \beta^2)} \]

Figure 5 shows a graphical depiction of the Newton-Raphson method for \( \gamma = 1.4 \), \( \nu = 35^\circ \) (0.6109 radians), for which \( M = 2.329 \). The initial guess is \( M = 1.1 \). We can see that the required Mach number is obtained within three iterations. In the web application we have fixed the initial guess at \( M = 1.1 \).
Figure 3  Graphical depiction of the Newton-Raphson method
4 Ranges of parameters

In the web application $\gamma$ is 1.4, which is the ratio of specific heats for the International Standard Atmosphere (Ref. [3]). When calculating the Prandtl-Meyer function from the Mach number, the Mach number may not be less than 1. When calculating the Mach number from the Prandtl-Meyer function, the Prandtl-Meyer function may not be less than $0^\circ$ or greater than $130^\circ$ (2.27 radians).
5 References