$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$E_b = \sigma T^4$$

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$$e^{i\pi} = -1$$

$$\frac{u}{u_\tau} = \frac{1}{\kappa}\ln\frac{yu_\tau}{\nu} + C$$

Equations of Fluid Flow

Momentum Equations in Cylindrical Coordinates

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Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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1 General vector form of the momentum equations

The momentum equations can be written in the general vector form:

$$\rho \frac{d\mathbf{V}}{dt} = \rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \,.\, \boldsymbol{\nabla}) \right] = -\boldsymbol{\nabla} P + \boldsymbol{\nabla} . \boldsymbol{\sigma} + \rho \mathbf{g} \qquad (1.1)$$

Where ρ is the density, $\mathbf{V}$ is the velocity vector, P is the pressure, μ is the dynamic viscosity and $\mathbf{g}$ is the acceleration due to gravity.

The term $d\mathbf{V}/dt$ is the material derivative, $\boldsymbol{\nabla}$ is the vector differential operator, and $\boldsymbol{\sigma}$ is the stress tensor σ_{ij}.

Equation (1.1) is the starting point for expressing the momentum equations in terms of cylindrical coordinates. In the next section we shall describe some of the mathematics of the vector differential operator $\boldsymbol{\nabla}$ in cylindrical coordinates. We shall use this mathematics to convert (1.1) into the momentum equations in terms of cylindrical coordinates. More information on the mathematics can be found in Refs. [1] and [2].

2 Mathematics of the vector differential operator

2.1 Divergence of a vector in cylindrical coordinates

In cylindrical coordinates, we define the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ in the radial, circumferential and axial directions, respectively. The unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are not constant but vary with θ. We can write the following relations:

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \qquad (2.1)$$

and

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \qquad (2.2)$$

The vector differential operator $\boldsymbol{\nabla}$ in cylindrical coordinates is

$$\boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \qquad (2.3)$$

We write a vector $\mathbf{V}$ in cylindrical coordinates as

$$\mathbf{V} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \qquad (2.4)$$

where v_r, v_θ and v_z are the components in the directions of the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$, respectively.

The divergence of a vector $\mathbf{V}$ in cylindrical coordinates is

$$\boldsymbol{\nabla} . \mathbf{V} = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}\right) . (v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z) \qquad (2.5)$$

When we evaluate (2.5), we cannot assume that the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are constants. We obtain

$$\boldsymbol{\nabla} . \mathbf{V} = \mathbf{e}_r \frac{\partial}{\partial r}(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z)$$

$$+ \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta}(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z)$$

$$\mathbf{e}_z \frac{\partial}{\partial z}(v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z)$$

or

$$\boldsymbol{\nabla} . \mathbf{V} = \mathbf{e}_r \frac{\partial (v_r \mathbf{e}_r)}{\partial r} + \mathbf{e}_r \frac{\partial (v_\theta \mathbf{e}_\theta)}{\partial r} + \mathbf{e}_r \frac{\partial (v_z \mathbf{e}_z)}{\partial r}$$

$$+ \frac{\mathbf{e}_\theta}{r}\frac{\partial (v_r \mathbf{e}_r)}{\partial \theta} + \frac{\mathbf{e}_\theta}{r}\frac{\partial (v_\theta \mathbf{e}_\theta)}{\partial \theta} + \frac{\mathbf{e}_\theta}{r}\frac{\partial (v_z \mathbf{e}_z)}{\partial \theta}$$

$$+ \mathbf{e}_z \frac{\partial (v_r \mathbf{e}_r)}{\partial z} + \mathbf{e}_z \frac{\partial (v_\theta \mathbf{e}_\theta)}{\partial z} + \mathbf{e}_z \frac{\partial (v_z \mathbf{e}_z)}{\partial z}$$

or

$$\nabla . \mathbf{V} = \mathbf{e}_r . \mathbf{e}_r \frac{\partial v_r}{\partial r} + \mathbf{e}_r v_r \frac{\partial \mathbf{e}_r}{\partial r} + \mathbf{e}_r . \mathbf{e}_\theta \frac{\partial v_\theta}{\partial r} + \mathbf{e}_r v_\theta \frac{\partial \mathbf{e}_\theta}{\partial r} + \mathbf{e}_r . \mathbf{e}_z \frac{\partial v_z}{\partial r} + \mathbf{e}_r v_z \frac{\partial \mathbf{e}_z}{\partial r}$$

$$+ \frac{\mathbf{e}_\theta . \mathbf{e}_r}{r} \frac{\partial v_r}{\partial \theta} + \frac{\mathbf{e}_\theta v_r}{r} \frac{\partial \mathbf{e}_r}{\partial \theta} + \frac{\mathbf{e}_\theta . \mathbf{e}_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\mathbf{e}_\theta v_\theta}{r} \frac{\partial \mathbf{e}_\theta}{\partial \theta} + \frac{\mathbf{e}_\theta . \mathbf{e}_z}{r} \frac{\partial v_z}{\partial \theta} + \frac{\mathbf{e}_\theta v_z}{r} \frac{\partial \mathbf{e}_z}{\partial \theta}$$

$$+ \mathbf{e}_z . \mathbf{e}_r \frac{\partial v_r}{\partial z} + \mathbf{e}_z v_r \frac{\partial \mathbf{e}_r}{\partial z} + \mathbf{e}_z . \mathbf{e}_\theta \frac{\partial v_\theta}{\partial z} + \mathbf{e}_z v_\theta \frac{\partial \mathbf{e}_\theta}{\partial z} + \mathbf{e}_z . \mathbf{e}_z \frac{\partial v_z}{\partial z} + \mathbf{e}_z v_z \frac{\partial \mathbf{e}_z}{\partial z} \qquad (2.6)$$

The products $\mathbf{e}_r . \mathbf{e}_r$, $\mathbf{e}_\theta . \mathbf{e}_\theta$ and $\mathbf{e}_z . \mathbf{e}_z$ are all equal to 1, and the products $\mathbf{e}_r . \mathbf{e}_\theta$, $\mathbf{e}_r . \mathbf{e}_z$, $\mathbf{e}_\theta . \mathbf{e}_r$, $\mathbf{e}_\theta . \mathbf{e}_z$, $\mathbf{e}_z . \mathbf{e}_r$, $\mathbf{e}_z . \mathbf{e}_\theta$ are all equal to zero. Apart from (2.1) and (2.2) the derivatives of the unit vectors are all equal to zero. Consequently, (2.6) becomes

$$\nabla . \mathbf{V} = \frac{\partial v_r}{\partial r} + 0 + 0 + 0 + 0 + 0$$

$$+ 0 + \frac{\mathbf{e}_\theta v_r}{r} \mathbf{e}_\theta + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\mathbf{e}_\theta v_\theta}{r} (-\mathbf{e}_r) + 0 + 0$$

$$+ 0 + 0 + 0 + 0 + \frac{\partial v_z}{\partial z} + 0$$

or

$$\nabla . \mathbf{V} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

or

$$\nabla . \mathbf{V} = \frac{1}{r} \frac{\partial (r v_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \qquad (2.7)$$

2.2 Gradient of a scalar in cylindrical coordinates

The gradient of a scaler P in cylindrical coordinates is

$$\nabla P = \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \right) P$$

$$= \mathbf{e}_r \frac{\partial P}{\partial r} + \frac{\mathbf{e}_\theta}{r} \frac{\partial P}{\partial \theta} + \mathbf{e}_z \frac{\partial P}{\partial z} \qquad (2.8)$$

2.3 Divergence of a second-order tensor in cylindrical coordinates

The divergence of a second-order tensor $\boldsymbol{\sigma} = [\sigma_{ij}]$ in cylindrical coordinates is

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \frac{\partial \sigma_{rr}}{\partial r}\mathbf{e}_r + \frac{\partial \sigma_{r\theta}}{\partial r}\mathbf{e}_\theta + \frac{\partial \sigma_{rz}}{\partial r}\mathbf{e}_z +$$

$$+\frac{1}{r}\left[\frac{\partial \sigma_{\theta r}}{\partial \theta} + \sigma_{rr} - \sigma_{\theta\theta}\right]\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \sigma_{r\theta} + \sigma_{\theta r}\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial \sigma_{\theta z}}{\partial \theta} + \sigma_{rz}\right]\mathbf{e}_z$$

$$+\frac{\partial \sigma_{zr}}{\partial z}\mathbf{e}_r + \frac{\partial \sigma_{z\theta}}{\partial z}\mathbf{e}_\theta + \frac{\partial \sigma_{zz}}{\partial z}\mathbf{e}_z \qquad (2.9)$$

(see Section 3 of Ref. [2]).

3 Material derivative in cylindrical coordinates

In a cylindrical coordinate system, the material derivative in the vector form of the momentum equations (1.1) is

$$\frac{d\mathbf{V}}{dt} = \frac{d}{dt}(v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z)$$

$$= \frac{d(v_r\mathbf{e}_r)}{dt} + \frac{d(v_\theta\mathbf{e}_\theta)}{dt} + \frac{d(v_z\mathbf{e}_z)}{dt}$$

$$= \mathbf{e}_r\frac{dv_r}{dt} + v_r\frac{d\mathbf{e}_r}{dt} + \mathbf{e}_\theta\frac{dv_\theta}{dt} + v_\theta\frac{d\mathbf{e}_\theta}{dt} + \mathbf{e}_z\frac{dv_z}{dt} + v_z\frac{d\mathbf{e}_z}{dt} \qquad (3.1)$$

Equations (2.1) and (2.2) can be arranged as follows:

$$d\mathbf{e}_r = \mathbf{e}_\theta d\theta \qquad (3.2)$$

$$d\mathbf{e}_\theta = -\mathbf{e}_r d\theta \qquad (3.3)$$

Also, we have

$$d\mathbf{e}_z = 0 \qquad (3.4)$$

Substituting (3.2), (3.3) and (3.4) into (3.1) gives

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_r\frac{dv_r}{dt} + v_r\frac{\mathbf{e}_\theta d\theta}{dt} + \mathbf{e}_\theta\frac{dv_\theta}{dt} + v_\theta\frac{-\mathbf{e}_r d\theta}{dt} + \mathbf{e}_z\frac{dv_z}{dt}$$

or

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_r\left(\frac{dv_r}{dt} - v_\theta\frac{d\theta}{dt}\right) + \mathbf{e}_\theta\left(\frac{dv_\theta}{dt} + v_r\frac{d\theta}{dt}\right) + \mathbf{e}_z\frac{dv_z}{dt} \qquad (3.5)$$

The term $d\theta/dt$ is the angular velocity ω, so we can write (3.5) as

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_r\left(\frac{dv_r}{dt} - v_\theta\omega\right) + \mathbf{e}_\theta\left(\frac{dv_\theta}{dt} + v_r\omega\right) + \mathbf{e}_z\frac{dv_z}{dt} \qquad (3.6)$$

The angular velocity ω is v_θ/r, so we can write (3.6) as

$$\frac{d\mathbf{V}}{dt} = \mathbf{e}_r\left(\frac{dv_r}{dt} - \frac{v_\theta^2}{r}\right) + \mathbf{e}_\theta\left(\frac{dv_\theta}{dt} + \frac{v_r v_\theta}{r}\right) + \mathbf{e}_z\frac{dv_z}{dt} \qquad (3.7)$$

We can now consider the individual vector components in (3.7).

Radial component of material derivative

The radial component of the material derivative is

$$\frac{dv_r}{dt} - \frac{v_\theta^2}{r} \qquad (3.8)$$

If we apply the chain rule to dv_r/dt then we obtain

$$\frac{dv_r}{dt} = \frac{\partial v_r}{\partial r}\frac{dr}{dt} + \frac{\partial v_r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial v_r}{\partial z}\frac{dz}{dt} + \frac{\partial v_r}{\partial t}\frac{dt}{dt}$$

$$= v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial t}$$

Hence, the radial component of the material derivative is

$$\frac{dv_r}{dt} - \frac{v_\theta^2}{r} = v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial t} - \frac{v_\theta^2}{r} \qquad (3.9)$$

Circumferential component of material derivative

The circumferential component of the material derivative is

$$\frac{dv_\theta}{dt} + \frac{v_r v_\theta}{r} \qquad (3.10)$$

If we apply the chain rule to dv_θ/dt then we obtain

$$\frac{dv_\theta}{dt} = \frac{\partial v_\theta}{\partial r}\frac{dr}{dt} + \frac{\partial v_\theta}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial v_\theta}{\partial z}\frac{dz}{dt} + \frac{\partial v_\theta}{\partial t}\frac{dt}{dt}$$

$$= v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{\partial v_\theta}{\partial t}$$

Hence, the circumferential component of the material derivative is

$$\frac{dv_\theta}{dt} + \frac{v_r v_\theta}{r} = v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{\partial v_\theta}{\partial t} + \frac{v_r v_\theta}{r} \qquad (3.11)$$

Axial component of material derivative

The axial component of the material derivative is

$$\frac{dv_z}{dt} \qquad (3.12)$$

If we apply the chain rule to dv_z/dt then we obtain

$$\frac{dv_z}{dt} = \frac{\partial v_z}{\partial r}\frac{dr}{dt} + \frac{\partial v_z}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial v_z}{\partial z}\frac{dz}{dt} + \frac{\partial v_z}{\partial t}\frac{dt}{dt}$$

$$= v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial t}$$

Hence, the axial component of the material derivative is

$$\frac{dv_z}{dt} = v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial t} \qquad (3.13)$$

4 Pressure gradient terms

The pressure gradient term $-\boldsymbol{\nabla}P$ in (1.1) can be written in its individual r, θ and z components using equation (2.8):

$$= -\left[\boldsymbol{e}_r \frac{\partial P}{\partial r} + \frac{\boldsymbol{e}_\theta}{r}\frac{\partial P}{\partial \theta} + \boldsymbol{e}_z \frac{\partial P}{\partial z}\right]$$

<u>r component</u>

$$-\frac{\partial P}{\partial r}$$

<u>θ component</u>

$$-\frac{1}{r}\frac{\partial P}{\partial \theta}$$

<u>z component</u>

$$-\frac{\partial P}{\partial z}$$

5 Viscous diffusion terms

In a cylindrical coordinate system the components of the strain rate tensor

$$\mathbf{e} = \begin{pmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\ e_{zr} & e_{z\theta} & e_{zz} \end{pmatrix}$$

are

$$e_{rr} = \frac{\partial v_r}{\partial r}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$$

$$e_{zz} = \frac{\partial v_z}{\partial z}$$

$$e_{r\theta} = e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$e_{rz} = e_{zr} = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

$$e_{\theta z} = e_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)$$

In a cylindrical coordinate system the stresses are related to the rates of strain in the fluid by the following equations:

$$\sigma_{rr} = 2\mu e_{rr} + \lambda \boldsymbol{\nabla} . \mathbf{V} = 2\mu \frac{\partial v_r}{\partial r} + \lambda \boldsymbol{\nabla} . \mathbf{V}$$

$$\sigma_{\theta\theta} = 2\mu e_{\theta\theta} + \lambda \boldsymbol{\nabla} . \mathbf{V} = 2\mu \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \lambda \boldsymbol{\nabla} . \mathbf{V}$$

$$\sigma_{zz} = 2\mu e_{zz} + \lambda \boldsymbol{\nabla} . \mathbf{V} = 2\mu \frac{\partial v_z}{\partial z} + \lambda \boldsymbol{\nabla} . \mathbf{V}$$

$$\sigma_{r\theta} = \sigma_{\theta r} = 2\mu e_{r\theta} = \mu \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$\sigma_{rz} = \sigma_{zr} = 2\mu e_{rz} = \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

$$\sigma_{\theta z} = \sigma_{z\theta} = 2\mu e_{\theta z} = \mu \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)$$

(see, for example, Ref. [3]).

In these equations μ is the *shear viscosity* or *first viscosity* and λ is the *volume viscosity* or *bulk viscosity*. The *second viscosity* ζ is defined by

$$\zeta = \lambda + \frac{2}{3}\mu$$

The second viscosity ζ is often assumed to be zero, making the volume viscosity λ equal to $-2\mu/3$.

The viscous diffusion terms in (1.1) are the divergence of a second-rank tensor, $\boldsymbol{\nabla} \,.\, \boldsymbol{\sigma}$. From (2.9) we have

$$\boldsymbol{\nabla} \,.\, \boldsymbol{\sigma} = \frac{\partial \sigma_{rr}}{\partial r}\mathbf{e}_r + \frac{\partial \sigma_{r\theta}}{\partial r}\mathbf{e}_\theta + \frac{\partial \sigma_{rz}}{\partial r}\mathbf{e}_z +$$

$$+\frac{1}{r}\left[\frac{\partial \sigma_{\theta r}}{\partial \theta} + \sigma_{rr} - \sigma_{\theta\theta}\right]\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \sigma_{r\theta} + \sigma_{\theta r}\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial \sigma_{\theta z}}{\partial \theta} + \sigma_{rz}\right]\mathbf{e}_z$$

$$+\frac{\partial \sigma_{zr}}{\partial z}\mathbf{e}_r + \frac{\partial \sigma_{z\theta}}{\partial z}\mathbf{e}_\theta + \frac{\partial \sigma_{zz}}{\partial z}\mathbf{e}_z \qquad (2.9)$$

If we substitute the stress relations above into (2.9) and separate the vector components, we obtain

<u>r component</u>

$$\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r} + \lambda\boldsymbol{\nabla} \,.\, \mathbf{V}\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\right] + \frac{1}{r}\left[2\mu\frac{\partial v_r}{\partial r} + \lambda\boldsymbol{\nabla} \,.\, \mathbf{V}\right] - \frac{1}{r}\left[2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \lambda\boldsymbol{\nabla} \,.\, \mathbf{V}\right]$$

$$+\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

This simplifies to

$$\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r} + \lambda\boldsymbol{\nabla} \,.\, \mathbf{V}\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\right] + \frac{2\mu}{r}\left[\frac{\partial v_r}{\partial r} - \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r}\right] + \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \qquad (5.1)$$

<u>θ component</u>

$$\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \lambda\boldsymbol{\nabla} \,.\, \mathbf{V}\right] + \frac{2\mu}{r}\left[\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right]$$

$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)\right] \qquad (5.2)$$

<u>z component</u>

$$\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)\right]+\frac{\mu}{r}\left[\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right]$$

$$+\frac{\partial}{\partial z}\left[2\mu\frac{\partial v_z}{\partial z}+\lambda\boldsymbol{\nabla}\,.\,\mathbf{V}\right]$$

This simplifies to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial v_z}{\partial z}+\lambda\boldsymbol{\nabla}\,.\,\mathbf{V}\right] \qquad (5.3)$$

6 Gravity force terms

The gravity force term $\rho\mathbf{g}$ in (1.1) can be written in terms of its individual r, θ and z components as follows:

$$= \rho[\boldsymbol{e}_r g_r + \boldsymbol{e}_\theta g_\theta + \boldsymbol{e}_z g_z] \quad (6.1)$$

<u>r component</u>

$$\rho g_r$$

<u>θ component</u>

$$\rho g_\theta$$

<u>z component</u>

$$\rho g_z$$

7 Momentum equations in cylindrical coordinates

We now write the momentum equations for the individual r, θ and z coordinate directions by gathering together the information on the components along each coordinate direction from the previous sections.

<u>r component</u>

$$\rho\left[\frac{dv_r}{dt}-\frac{v_\theta^2}{r}\right]=\rho\left[v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+v_z\frac{\partial v_r}{\partial z}+\frac{\partial v_r}{\partial t}-\frac{v_\theta^2}{r}\right]=-\frac{\partial P}{\partial r}+\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r}+\lambda\boldsymbol{\nabla}.\mathbf{V}\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]+\frac{2\mu}{r}\left[\frac{\partial v_r}{\partial r}-\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}-\frac{v_r}{r}\right]+\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)+\rho g_r \qquad (7.1)$$

<u>θ component</u>

$$\rho\left[\frac{dv_\theta}{dt}+\frac{v_r v_\theta}{r}\right]=\rho\left[v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{\partial v_\theta}{\partial t}+\frac{v_r v_\theta}{r}\right]=-\frac{1}{r}\frac{\partial P}{\partial\theta}$$

$$+\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)+\lambda\boldsymbol{\nabla}.\mathbf{V}\right]$$

$$+\frac{2\mu}{r}\left[\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right)\right]+\rho g_\theta \qquad (7.2)$$

<u>z component</u>

$$\rho\frac{dv_z}{dt}=\rho\left[v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}+\frac{\partial v_z}{\partial t}\right]=-\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial v_z}{\partial z}+\lambda\boldsymbol{\nabla}.\mathbf{V}\right]+\rho g_z \qquad (7.3)$$

In cylindrical coordinates the continuity equation is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta}+\frac{\partial(\rho v_z)}{\partial z}=0 \qquad (7.4)$$

If we multiply the continuity equation by the velocity component v_r, v_θ or v_z and add the result to the left-hand side of the corresponding momentum equation then we obtain

<u>r component</u>

$$\frac{1}{r}\frac{\partial(r\rho v_r^2)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta v_r)}{\partial\theta}+\frac{\partial(\rho v_z v_r)}{\partial z}+\frac{\partial(\rho v_r)}{\partial t}-\frac{\rho v_\theta^2}{r}=-\frac{\partial P}{\partial r}+\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r}+\lambda\boldsymbol{\nabla}.\mathbf{V}\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]+\frac{2\mu}{r}\left[\frac{\partial v_r}{\partial r}-\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}-\frac{v_r}{r}\right]+\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)+\rho g_r \qquad (7.5)$$

<u>θ component</u>

$$\frac{1}{r}\frac{\partial(r\rho v_r v_\theta)}{\partial r}+\frac{1}{r}\frac{\partial\left(\rho v_\theta^2\right)}{\partial\theta}+\frac{\partial(\rho v_z v_\theta)}{\partial z}+\frac{\partial(\rho v_\theta)}{\partial t}+\frac{\rho v_r v_\theta}{r}=-\frac{1}{r}\frac{\partial P}{\partial\theta}$$

$$+\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)+\lambda\boldsymbol{\nabla}.\mathbf{V}\right]$$

$$+\frac{2\mu}{r}\left[\frac{1}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right)\right]+\rho g_\theta \quad (7.6)$$

<u>z component</u>

$$\frac{1}{r}\frac{\partial(r\rho v_r v_z)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta v_z)}{\partial\theta}+\frac{\partial(\rho v_z^2)}{\partial z}+\frac{\partial(\rho v_z)}{\partial t}=-\frac{\partial P}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]$$

$$+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\mu\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial v_\theta}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial v_z}{\partial z}+\lambda\boldsymbol{\nabla}.\mathbf{V}\right]+\rho g_z \quad (7.7)$$

This form of the momentum equation is often referred to as the *conservative* form.

8 References

1. M. R. Spiegel, *Vector Analysis and an Introduction to Tensor Analysis*, Schaum's Outline Series, McGraw-Hill, 1959.
2. Wikipedia, Tensor derivative (continuum mechanics), 2022, https://en.wikipedia.org/wiki/Tensor_derivative_(continuum_mechanics).
3. W. F. Hughes and E. W. Gaylord, *Basic Equations of Engineering Science*, Schaum's Outline Series, McGraw-Hill, 1964.
4. K. N. Atkinson, *Equations of Fluid Flow, Continuity Equation in Cartesian and Cylindrical Coordinates, Theory Guide*, Atkinson Science Limited, 2020.